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Quantum fluctuations of the center-of-mass and relative parameters of NLS breathers

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We study quantum fluctuations of macroscopic parameters of an NLS breathers, i.e., the second-order soliton solution of the nonlinear Schrödinger equation. Uncertainty relations for the parameters are derived and compared to similar relations for fundamental solitons. We compare two models for the state of the quantum field of fluctuations surrounding the classical field of the Bose-Einstein condensate: a conventionally used, computationally convenient "white noise", and a correlated noise which assumes that the breather has been created from a fundamental soliton, by means of the application of the factor-of-four quench of the nonlinearity strength. Theoretical methods used in the work are well suited for a large number of particles, N. We thus confirm the possibility of experimental observation of macroscopic quantum fluctuations, which is suggested by an extrapolation to large N of recently reported low-N Bethe-ansatz results [Phys. Rev. Lett. **119**, 220401 (2017)].

Introduction and models. — The nonlinear Schrödinger (NLS) equation plays a fundamental role in many areas of physics, from Langmuir waves in plasmas [1] to the propagation of optical signals in nonlinear waveguides [2–6]. With recent developments in the creation of Bose-Einstein condensates (BECs) in ultracold atomic gases, the corresponding form of the NLS equation, which provides the mean-field approximation for the rarefied BEC, namely the Gross-Pitaevskii (GP) equation, has drawn a great deal of interest. Experimentally, bright solitons predicted by the GP equation were observed in ultracold ^{7}Li [7–9] and ^{85}Rb [10, 11] gases, in the quasi-one-dimensional (1D) regime imposed by a cigar-shaped potential trap. Because the GP equation is a mean-field model, and thus does not include quantum fluctuations, one needs to include quantum many-body effects to achieve a more realistic description of the system. The simplest approach is to employ the linearization method first proposed by Bogoliubov [12] in the context of superfluid quantum liquids. For more than two decades, this method has been successfully used to describe excitations in BECs [13–16].

We emphasize the focusing nonlinearity corresponding to the GP equation that describes a Bose gas with attractive interactions between atoms. The NLS equation belongs to a class of integrable systems [17–19], thus having infinitely many dynamical invariants and infinitely many species of soliton solutions. The simplest one, the fundamental soliton, is a localized stationary mode, which can move with an arbitrary velocity. It is often referred as the *bright soliton*, due to its origin in the context of nonlinear optics. The next-order solution, i.e., a 2-soliton wave function, may be found by means of an inverse-scattering-transform method [20]. It is localized in space and oscillates in time, being commonly called a *breather*. This solution may be interpreted as a nonlinear bound state of two fundamental solitons with a $1:3$ mass ratio and exactly zero interaction energy [18, 21]. The 2-soliton breather can be created by a factor-of-four quench applied to the nonlinearity strength, starting from a single fundamental soliton, as was predicted long ago in the analytical form [20], and recently demonstrated experimentally in a BEC soliton [22]. At the mean-field level, the relative velocity of the fundamental solitons whose bound state forms the breather is identically equal to zero, regardless of how hot the state of the center of mass (COM) of the "mother" soliton was. Thus, if the breather spontaneously splits in free space, intrinsic quantum fluctuations are expected to be the *only* cause of the fission (at the mean-field level, controllable splitting of the breather can be induced by a local linear or nonlinear potential [23]). This paves a way for a potential experimental observation of the splitting as a macroscopic manifestation of *quantum fluctuations in a macroscopic object*, which may take place under standard mean-field experimental conditions. An estimate for the splitting time was already obtained in Ref. [24], for a low number of particles, using the exact many-body solution provided by the Bethe ansatz (BA). A related work [25] studied another beyond-mean-field effect: developing of decoherence between the two constituent solitons. While an experimen-

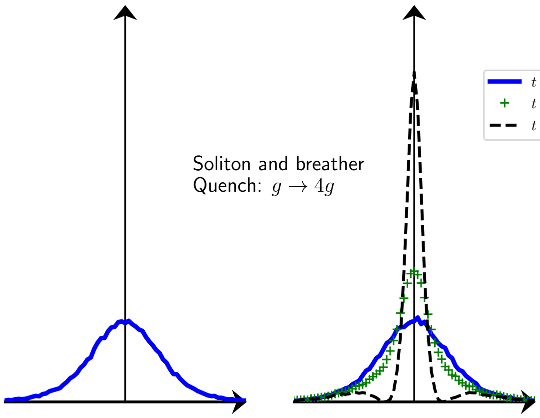

FIG. 1. A schematic representation of the fundamental "mother soliton", as the vacuum state with inherent correlated quantum noise (the left panel), transformed into the breather by means of the interaction quench (the right panel).

tal observation of quantum behavior of the COM of a (macro/meso)-scopic soliton (e.g., the effects analyzed in [26–28]) remains elusive, several groups have been making progress towards this goal [22, 29].

In this work, in order to determine the evolution of the quantum fluctuations, we employ the linearization method based on the Bogoliubov theory, that was first used for fundamental solitons in Refs. [30–32] in the context of the signal transmission in optical waveguides. Later, Yeang [33] had extended the analysis for the COM parameters of a breather. The main focus of the present work is on quantum fluctuations of relative parameters of the breather. We calculate initial variances of the quantum fluctuations of the parameters, that allow us to produce uncertainty relations for the relative amplitude ratio and relative phase, as well as relative coordinate and momentum operators. The obtained relations are compared to the uncertainty relations for operators representing parameters of the fundamental soliton [31, 32].

Below, we compare two models for the halo of quantum fluctuations surrounding the mean-field states of the atomic BEC: a conventionally used, computationally convenient "white noise" [25, 30, 31] of vacuum fluctuations, and a correlated noise, assuming that the breather has been created from a fluctuating fundamental soliton, by means of the above-mentioned factor-of-four quench applied to the nonlinearity coefficient, as schematically shown in Fig. 1.

Thus, we consider a gas of Bose atoms with the s-wave scattering length $a_{\rm sc} < 0$ in an elongated trap with transverse frequency $\omega_\perp$ [7, 8, 34]. The scattering length can be tuned by an external magnetic field, using a Feshbach resonance [35]. Whenever the kinetic energy is less than $\hbar\omega_\perp$, the atoms may be considered as 1D particles with the attractive zero-range interaction between them, of strength $-g = 2\hbar\omega_\perp a_{\rm sc}$ [36]. The 1D gas is described by the quantum (Heisenberg's) NLS equation,

$$i\hbar\frac{\partial\hat\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\hat\Psi(x,t)}{\partial x^2} - g\hat\Psi^\dagger(x,t)\hat\Psi(x,t)\hat\Psi(x,t), \tag{1}$$

where m is the atomic mass. The creation and annihilation quantum-field operators, $\hat\Psi^\dagger$ and $\hat\Psi$, obey the standard bosonic commutation relations.

In the linearization method, following the Bogoliubov approach, the quantum field $\hat\Psi$ is expressed a $\hat\Psi(x,t) = \sqrt{N}\Psi_0(x,t) + \delta\hat\psi(x,t)$, where N is the number of atoms. Here the mean field $\sqrt{N}\Psi_0(x,t)$, a solution of the classical NLS equation (1), represents the condensed part of the Bose gas. Operator $\delta\hat\psi(x,t)$ and its Hermitian conjugate $\delta\hat\psi(x,t)^\dagger$ represent quantum fluctuations and also obey the standard bosonic commutation relations. In the framework of the linearization method, $\delta\hat\psi$ is assumed to be small in comparison to the condensate wave function, $\sqrt{N}\Psi_0$, allowing one to linearize Eq. (1) with respect to $\delta\hat\psi$:

$$i\hbar\frac{\partial\delta\hat\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\delta\hat\psi}{\partial x^2} - 2gN|\Psi_0|^2\delta\hat\psi - gN\Psi_0^2\delta\hat\psi^\dagger. \tag{2}$$

Applying the linearization method to NLS breathers, we use the Gordon's solution of the NLS equation [37] for two solitons with numbers of atoms N_1 and N_2, as shown in detail in the supplement [38]. This solution depends on 8 parameters. Four of them are related to the COM motion: the number of atoms $N = N_1 + N_2$, overall phase Θ, COM velocity V, and COM coordinate B. The remaining four parameters are the relative velocity v of the constituent solitons, initial distance between them, b, initial relative phase difference, θ, and mass/amplitude difference, $n = (N_2 - N_1)$. The particular case of $n = \pm\frac{1}{2}N$, $v = b = \theta = 0$, corresponds to the breather solution. In the COM frame of reference ($V = 0$), the breather remains localized and oscillates with period $T_{\rm br} = 32\pi\hbar^3/(mg^2N^2)$. On the other hand, the fundamental soliton is obtained for $n = \pm N$, $v = b = \theta = 0$.

The quantum correction to the two-soliton solution is

$$\delta\hat\psi = \sum_\chi f_\chi(x,t)\Delta\hat\chi_0 + \hat\psi_{\rm cont}(x,t), \tag{3}$$

where χ is one of the 8 parameters (N, Θ, V, B, n, θ, v, and b), and $f_\chi(x,t) = \partial(\sqrt{N}\Psi_0)/\partial\chi$ are derivatives of the mean-field solution with respect to them. Then, the sum in Eq. (3) is an exact operator solution of linearized equation (2). The Hermitian operators $\Delta\hat\chi_0$, introduced in Refs. [30, 31], may be considered as quantum fluctuations of parameters χ at $t = 0$, since they have the same effect on the density as classical fluctuations of the mean-field parameters (see details in the supplement [38]). The set of 8 parameters χ is related to breaking of the $U(1)$ and translational symmetries of the underlying Hamiltonian, hence they are related to the Goldstone and "lost" modes, in the framework of the Bogoliubov-de Gennes description [28, 39, 40]. Operator $\hat{\psi}_{\rm cont}$ describes continuum fluctuations which are analyzed in Ref. [41] for the fundamental soliton. In this work we assume orthogonality of the breather's continuum fluctuations $\hat{\psi}_{\rm cont}$ to the discrete-expansion modes, leaving a rigorous proof of this fact for subsequent work. Indeed, there are good reasons for this conjecture: Firstly, in the context of nonlinear optics (see, e.g., Refs. [31, 33]) it is supported by the fact that, in the limit of $t \to \infty$, continuum modes completely disperse out, hence the orthogonality condition definitely holds. Secondly, careful construction of Bogoliubov eigenstates ensures the orthogonality of the Goldstone and continuum modes by default [42].

Operators $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$ may be interpreted as creation/annihilation operators of the quantum fluctuations. To properly define the action of the operators, one has to specify the physical nature of the vacuum state. The breather solution is created as follows. (i) The state is initialized as a bright *mother soliton*. (ii) Then, the sudden quench of the interaction strength, namely, its fourfold increase [17, 20, 43], transforms the mother soliton into a 2-soliton breather solution, which is the subject of the work.

The mother-soliton input defines the vacuum state of the quantum-fluctuation operators around the breather. Below we address two different approaches for incorporating properties of the mother-soliton state into the linearization scheme.

The white-noise vacuum. — The most widespread approach to introduce the vacuum state for $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$ operators is to consider the "white noise" vacuum, as is standard in nonlinear optics [6, 31–33, 41]. This approach considers vacuum fluctuations as uncorrelated random noise. In atomic physics such formulation is also used [25], and has the following interpretation: the mother soliton is a Hartree product of the non-interacting single-particle wave functions, that all have the shape of the mother soliton. Thus, the expectation values of the fluctuation creation and annihilation operators $\delta\hat{\psi}^\dagger(x,0)$ and $\delta\hat{\psi}(x,0)$ are

$$\langle \delta\hat{\psi}(x,0)\delta\hat{\psi}(x',0)\rangle = \langle \delta\hat{\psi}^\dagger(x,0)\delta\hat{\psi}^\dagger(x',0)\rangle = 0, \tag{4a}$$

$$\langle \delta\hat{\psi}^\dagger(x,0)\delta\hat{\psi}(x',0)\rangle = 0, \tag{4b}$$

$$\langle \delta\hat{\psi}(x,0)\delta\hat{\psi}^\dagger(x',0)\rangle = \delta(x-x'), \tag{4c}$$

where the averaging $\langle ...\rangle$ is assumed over the vacuum state. Thus, only one product of the operators $\delta\hat{\psi}^\dagger(x,0)$ and $\delta\hat{\psi}(x,0)$ yields a nonzero contribution. At $t=0$, quantum fluctuations of 8 parameters $\Delta\hat{\chi}_0$ can be expressed in terms of overlaps of derivatives $f_\chi(x,t)$ as

$$\langle \Delta\hat{\chi}_0^2\rangle = \frac{1}{4C^2_{\tilde{\chi}\chi}}\int_{-\infty}^{\infty} {\rm d}x |f_{\tilde{\chi}}(x,0)|^2 \tag{5}$$

(see further details in supplement [38]), where the 8 parameters are grouped to four canonically conjugate pairs, *viz.*, (N, Θ), (V, B), (n, θ), and (v, b). Derivatives of the Gordon's solution and overlap integrals are evaluated analytically (using Wolfram Mathematica). The evaluated fluctuations are given in Table I, where scales of the length and velocity, $\bar{x} = \hbar^2/(mg)$ and $\bar{v} = g/\hbar$, are used. For ^{7}Li atoms with $m = 7$ AMU, $\omega_\perp = 254 \times 2\pi$ Hz, and $a_{\rm sc} = -4a_0$, where a_0 is the Bohr radius, we have $\bar{x} \approx 1.34$ cm and $\bar{v} \approx 6.75 \times 10^{-5}$cm/s, while the breather's oscillation period is $T_{\rm br} \approx 4 \times 10^6/N^2$ s.

	Number	Phase	velocity	coordinate
COM	N	$\frac{12+\pi^2}{36N}$	$\frac{N\bar{v}^2}{192}$	$\frac{16\pi^2\bar{x}^2}{3N^3}$
Rel.	$N/5$	$\frac{4(420+23\pi^2)}{315N}$	$\frac{23N\bar{v}^2}{420}$	$\frac{256\pi^2\bar{x}^2}{15N^3}$

TABLE I. Initial values of the quantum fluctuations $\langle\Delta\hat{\chi}_0^2\rangle$ of the COM (center-of-mass) and relative parameters of the breather, obtained from the *white-noise* vacuum states.

Now we can compare the uncertainty expressions with the standard (Heisenberg's) quantum limit:

$$\langle\Delta\hat{N}_0^2\rangle\langle\Delta\hat{\Theta}_0^2\rangle \approx 0.608 > 0.25 \tag{6a}$$

$$N^2m^2\langle\Delta\hat{V}_0^2\rangle\langle\Delta\hat{B}_0^2\rangle/\hbar^2 \approx 0.274 > 0.25, \tag{6b}$$

$$\langle\Delta\hat{n}_0^2\rangle\langle\Delta\hat{\theta}_0^2\rangle \approx 1.64, \tag{6c}$$

$$N^2(3m/16)^2\langle\Delta\hat{v}_0^2\rangle\langle\Delta\hat{b}_0^2\rangle/\hbar^2 \approx 0.3243, \tag{6d}$$

where we take the uncertainty of the COM and relative momenta as $N^2m^2\langle\Delta\hat{V}_0^2\rangle$ and $N^2(3m/16)^2\langle\Delta\hat{v}_0^2\rangle$, respectively. Note that the uncertainty value for the conjugate pair of the relative momentum, $3mNv/16$, and relative distance, b, is $\approx 20\%$ larger than that for COM momentum-position pair. One can also evaluate averages of the cross-products of the operators, using the formula similar to Eq.(5), see details in supplement [38]. The non-vanishing values

$$\begin{aligned}\langle\Delta\hat{N}_0\Delta\Theta_0\rangle = i/2, \quad \langle\Delta\hat{B}_0\Delta\hat{V}_0\rangle = i\hbar/(2Nm)\\ \langle\Delta\hat{n}_0\Delta\hat{\theta}_0\rangle = i, \quad \langle\Delta\hat{b}_0\Delta\hat{v}_0\rangle = 8i\hbar/(3Nm)\end{aligned} \tag{7}$$

are purely imaginary due to the properties of the modes $f_{\bar{\chi}}$, and $\langle\Delta\hat{\chi}_0\Delta\hat{\chi}_0'\rangle = -\langle\Delta\hat{\chi}_0'\Delta\hat{\chi}_0\rangle$ due to the hermiticity.

Note that $\sqrt{\langle\Delta\hat{N}_0^2\rangle\langle\Delta\hat{\Theta}_0^2\rangle} \approx 0.78$, $\sqrt{\langle\Delta\hat{V}_0^2\rangle\langle\Delta\hat{B}_0^2\rangle} \approx 2.1\hbar/(Nm)$, $\sqrt{\langle\Delta\hat{n}_0^2\rangle\langle\Delta\hat{\theta}_0^2\rangle} \approx 1.3$, and $\sqrt{\langle\Delta\hat{v}_0^2\rangle\langle\Delta\hat{b}_0^2\rangle} \approx 3\hbar/(Nm)$. Then, cross term $\langle\Delta\hat{B}_0\Delta\hat{V}_0\rangle$ may be neglected, while others are non-negligible.

Contributions from mother-soliton's continuum fluctuations. — The predictions for fluctuation of the breather's parameter are significantly changed if field fluctuations of the mother (pre-quench) soliton are included. In contrast to the white-noise vacuum case, we cannot keep only one of the fluctuating operators products, $\delta\hat{\psi}(x)\delta\hat{\psi}^\dagger(x')$, thus the correlated-quantum-noise vacuum leads to different expectation values. Quantum fluctuations of the mother soliton can be separated into discrete and continuum parts [28, 39, 40]. Further, expectation values of the continuum creation/annihilation operator products can be calculated, using known exact expressions [28, 44] for the Bogoliubov modes of the mother soliton (for details see supplement [38]). Discrete fluctuations of the mother soliton are determined by derivatives of the mean field with respect to the soliton's parameters. They coincide with the breather's COM fluctuations, as the soliton's and breather's mean fields are the same at $t = 0$. Then, fluctuations of the discrete parameters of the mother soliton are decoupled from the relative degrees of freedom of the breather, hence they do not affect the corresponding variances (see details in supplement [38]). They are, of course, important to evaluate uncertainties of the COM's degrees of freedom of the breather, but as they are defined by parameters of the experiment that creates the mother soliton, we do not consider them in this work. Note also that, due to phase-diffusion effects [39, 40], the contribution from the fluctuations of the discrete parameters of the mother soliton would have to depend on time elapsing from the creation of the mother soliton until the application interaction quench.

In Table II we compare numerical factors for initial variances of the relative parameters for different vacuum states. Due to the complicated form of the expressions, the variances for the correlated-noise vacuum could not be obtained in a closed analytical form, therefore they were evaluated numerically. The difference, while not being enormous, is evident and it clearly may affect dynamics of the breather. Note that the correlated vacuum does not affect the expectation values of the cross-products of the operators, hence they are the same as for the white-noise vacuum, see Eq. (7).

Noise	$\langle\Delta\hat{n}_0^2\rangle$	$\langle\Delta\hat{\theta}_0^2\rangle$	$\langle\Delta\hat{v}_0^2\rangle$	$\langle\Delta\hat{b}_0^2\rangle$
White	$0.2N$	$8.22/N$	$0.0548N\bar{v}^2$	$168\bar{x}^2/N^3$
Correlated	$0.3N$	$6.26/N$	$0.0429N\bar{v}^2$	$198\bar{x}^2/N^3$

TABLE II. Initial quantum fluctuations of relative parameters of the breather, for the white-noise and pre-quench correlated-vacuum states.

In Fig. 2 we display the evolution of the variances of quantum operators of the relative parameters of the breather, and compare the results for the white-noise and pre-quench correlated-noise vacuum states.

Eventually, these results allow us to evaluate time required for a breather to dissociate under the action of the fluctuations of the relative velocity in realistic experimental conditions. The fission time was previously evaluated in Ref. [24] on the basis of the BA solution, yielding $\tau_{\mathrm{BA}} \simeq 3$ s. Here, we consider a quasi-1D gas of $N = 3 \times 10^3$ ^{7}Li atoms with the trapping and scattering parameters mentioned above. We assume that the splitting of the breather can be detected once the constituent solitons are separated by a distance comparable to the

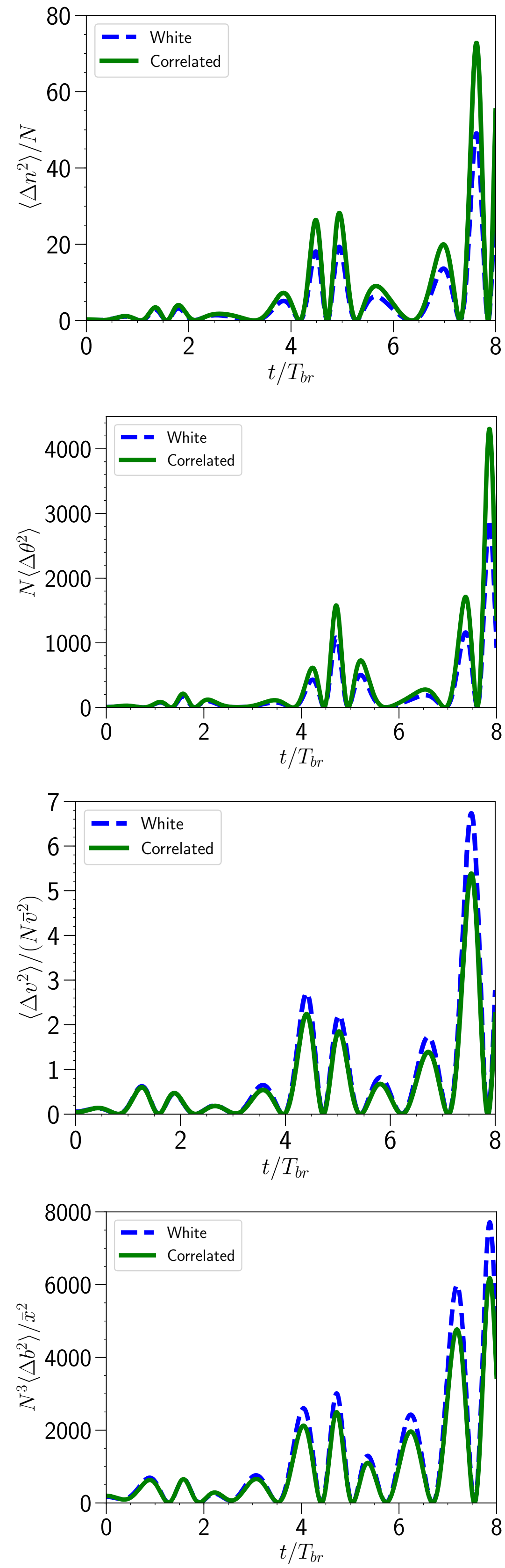

FIG. 2. Variances of fluctuations of the relative parameters of the breather, as a function of time (from top to bottom): the number of atoms $\langle\Delta\hat{n}^2(t)\rangle$, phase $\langle\Delta\hat{\theta}^2(t)\rangle$, velocity $\langle\Delta\hat{v}^2(t)\rangle$, and position $\langle\Delta\hat{b}^2(t)\rangle$, as found for the white-noise vacuum state (blue dashed lines) and pre-quench correlated vacuum state (green solid lines).

breather's width, which is given by $l_{\mathrm{br}} = 8\hbar^2/(mgN) \approx 36$ μm [17]. Thus, the dissociation time may be evaluated as $\tau = l_{\mathrm{br}}/\sqrt{\langle \Delta v_0^2 \rangle}$. The final result can be obtained for fluctuations obtained with different vacuum states. Using Table II, we obtain $\tau_{\mathrm{white}} \approx 4.16$ s, and $\tau_{\mathrm{corr}} \approx 4.7$ s. Thus, the inclusion of the continuum fluctuations of the mother soliton increases the dissociation time by ≈ 0.55 s, which may be a significant difference for the experiment. Note that the estimate in Ref. [24] used a different definition of the dissociation time; using the present definition, the *ab initio* BA calculations leads to $\tau \approx 5.18$ s.

Conclusions. — In this work we consider quantum fluctuations of a NLS breather, composed of two fundamental solitons with amplitude ratio 1 : 3. In an experiment, the breather is made from a fundamental (mother) soliton by a factor-of-four quench applied to the interaction strength. The Bogoliubov linearization approach allows one to estimate variances of the quantum fluctuations of the breather's discrete parameters, which include its COM (center-of-mass) characteristics and relative degrees of freedom. To evaluate expectation values, one needs to properly define the vacuum state of the system. We consider two cases: a popular and computationally convenient uncorrelated quantum noise, also referred as a "white noise", and a state with correlated quantum noise produced by the quench, that takes into account pre-quench quantum fluctuations of the mother soliton. The comparison shows that the correlated noise noticeably changes initial values and the evolution of the variances. Our analysis provides estimates for the time of the dissociation of the breather, induced by the quantum fluctuations. The estimates yield, for realistic experimental parameters, $\tau_{\mathrm{corr}} \approx 4.7$ s for the correlated noise vacuum; this value is approximately 0.55 s larger than the corresponding result for the uncorrelated noise. It is also rather close to a similar estimate extrapolated from an *ab initio* result of Ref. [24]. The obtained results indicate feasibility of experimental observation of a manifestation of quantum fluctuations of macroscopic degrees of freedom—in particular, the relative velocity of the two initially bound solitons. Note also that the closeness of the uncertainty relation (6d) to the Heisenberg's lower limit for the position-momentum uncertainty indicates that the state of the relative motion is *provably* a macroscopically quantum one: if it were occupying a large phase-space area, it could—while remaining formally a pure state—become chaotic in the course of the subsequent quantum evolution, while it does not do it.

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Supplemental material for: Quantum fluctuations of the center-of-mass and relative parameters of NLS breather

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Numbers of equations and figures in the Supplemental material start with S. References to equations and figures in the Letter do not contain S.

THE GORDON TWO-SOLITON SOLUTION

The mean -field $\Psi_0(x,t)$ is the solution of the classical NLS (GP) equation,

$$i\frac{\partial\Psi_0}{\partial t}=-\frac{1}{2}\frac{\partial^2\Psi_0}{\partial x^2}-N|\Psi_0|^2\Psi_0, \tag{S-1}$$

where the coordinates and time are measured in the units $\bar{x}=\hbar^2/(mg)$ and $\bar{t}=\hbar^3/(mg^2)$, respectively. For two solitons containing N_1 and N_2 atoms, the solution was obtained in [37],

$$\begin{aligned}\Psi_0(x,t)&=\frac{\sqrt{N}}{2}\left(\Phi_+(x-B-Vt,t)+\Phi_-(x-B-Vt,t)\right)\exp\left(i\phi(x-B-Vt)+iVx-iV^2t/2+i\Theta\right),\\ \Phi_\pm(x,t)&=e^{\pm i\varphi}\frac{(1\pm n/N)\left(\frac{Nn\pm4v^2}{N^2}\right)\cosh(\frac{Nx}{4}\mp z)-i((n/N)^2-1)\frac{2v}{N}\sinh(\frac{Nx}{4}\mp z)}{(1-(n/N)^2)\cos(2\varphi)+\left(\frac{n^2+4v^2}{N^2}\right)\cosh(\frac{N}{2}x)+\left(\frac{4v^2}{N^2}+1\right)\cosh(2z)},\end{aligned} \tag{S-2}$$

where $\phi(x)=\frac{1}{2}\left((n/N)^2+1\right)\left(\frac{N^2-4v^2}{16}\right)t-\frac{n}{N}\frac{v}{2}x$, $z=\frac{N}{4}\left(\frac{1}{2}\left(1-(n/N)^2\right)(b-tv)+nx/N\right)$, and $\varphi=\frac{n}{4N}v^2t+n\frac{N}{16}t+\frac{1}{2}vx+\frac{\theta}{2}$. This solution depends on 8 parameters, namely, the number of atoms $N=N_1+N_2$, phase Θ, COM velocity V, initial COM coordinate B, relative velocity of the constituent solitons v, initial distance between the solitons b, relative phase difference θ, and the amplitude difference, $n=(N_2-N_1)$. The velocities are measured in the units $\bar{v}=g/\hbar$. The solution is normalized such that $\int_{-\infty}^{+\infty}\mathrm{d}x|\Psi_0(x,t)|^2=1$.

THE RELATION BETWEEN HERMITIAN OPERATORS $\Delta\hat{\chi}_0$ AND FLUCTUATIONS OF PARAMETERS χ OF THE MEAN-FIELD SOLUTION

To derive the relation, we use the fact that the ensemble average of the mean-field-solution density matrix should give the same result as the expectation value of the density operator constructed from the quantum field:

$$\overline{\Psi_0^*(x,t;\chi)\Psi(x,t;\chi)}=\langle\hat{\Psi}^\dagger(x,t)\hat{\Psi}(x,t)\rangle \tag{S-3}$$

To calculate the density, we need to know mean field $\tilde{\Psi}_0$ that takes into account fluctuation densities according to the Hartree-Fock-Bogoliubov equation [45, 46]:

$$i\frac{\partial\tilde{\Psi}_0}{\partial t}=-\frac{1}{2}\frac{\partial^2\tilde{\Psi}_0}{\partial x^2}-N|\tilde{\Psi}_0|^2\tilde{\Psi}_0-2\left\langle\delta\hat{\psi}^\dagger\delta\hat{\psi}\right\rangle\tilde{\Psi}_0-\left\langle\delta\hat{\psi}\delta\hat{\psi}\right\rangle\tilde{\Psi}_0^*. \tag{S-4}$$

It can be approximated by the sum $\tilde{\Psi}_0\approx\Psi_0+\Psi_2$, where Ψ_0 is the solution of the NLS equation (S-1), and Ψ_2 is the correction which satisfies the linear driven equation,

$$i\frac{\partial\Psi_2}{\partial t}=-\frac{1}{2}\frac{\partial^2\Psi_2}{\partial x^2}-2N|\Psi_0|^2\Psi_2-N\Psi_0^2\Psi_2^*-\sum_{\chi,\chi'}\left(2f_\chi^*\Psi_0+f_\chi\Psi_0^*\right)f_{\chi'}\left\langle\Delta\hat{\chi}\Delta\hat{\chi}'\right\rangle, \tag{S-5}$$

where we use expansion $\delta\hat{\psi}=\sum_\chi f_\chi\Delta\hat{\chi}$, with $f_\chi(x,t)=\partial(\sqrt{N}\Psi_0)/\partial\chi$. Thus, Ψ_2 can be expressed as

$$\Psi_2=\frac{1}{2N}\sum_{\chi,\chi'}\frac{\partial^2}{\partial\chi\partial\chi'}\Psi_0\left\langle\Delta\hat{\chi}\Delta\hat{\chi}'\right\rangle, \tag{S-6}$$

as the second derivative of Ψ_0 satisfies the differential equation with the same homogeneous part as in Eq. (S-5). Then, using the expansion of the field operator, $\hat{\Psi}(x,t) = \sqrt{N}\tilde{\Psi}_0(x,t) + \delta\hat{\psi}(x,t)$, we can calculate the density

$$\left\langle \hat{\Psi}^\dagger \hat{\Psi} \right\rangle \approx N\Psi_0^*\Psi_0 + \frac{1}{2}N \sum_{\chi,\chi'} \left(\Psi_0^* \frac{\partial^2}{\partial\chi\partial\chi'}\Psi_0 + \Psi_0 \frac{\partial^2}{\partial\chi\partial\chi'}\Psi_0^* + 2\frac{\partial\Psi_0^*}{\partial\chi}\frac{\partial\Psi_0}{\partial\chi'} \right) \langle \Delta\hat{\chi}\Delta\hat{\chi}' \rangle$$
$$= N\Psi_0^*\Psi_0 + \frac{1}{2}N \sum_{\chi,\chi'} \frac{\partial^2}{\partial\chi\partial\chi'} \left(\Psi_0^*\Psi_0\right) \langle \Delta\hat{\chi}\Delta\hat{\chi}' \rangle . \tag{S-7}$$

On the other hand, we calculate the ensemble average of the classical field solution (S-2) that depends on fluctuating parameters χ, using the Taylor expansion

$$\Psi_0(x,t;\chi) = \Psi_0(x,t;\chi_0) + \sum_\chi \left(\frac{\partial\Psi_0}{\partial\chi}\right)_{\chi=\chi_0} \delta\chi + \frac{1}{2}\sum_{\chi,\chi'} \left(\frac{\partial^2}{\partial\chi\partial\chi'}\Psi_0\right)_{\chi=\chi_0} \delta\chi\delta\chi' + \ldots, \tag{S-8}$$

where $\delta\chi = \chi - \chi_0$. Then, the mean-field density $N\Psi_0^*\Psi_0$, averaged over classical fluctuations of parameters $\delta\chi$ is (linear terms vanish here)

$$N\overline{\Psi_0^*\Psi_0} \approx N\Psi_0^*\Psi_0 + \frac{1}{2}N \sum_{\chi,\chi'} \frac{\partial^2}{\partial\chi\partial\chi'} \left(\Psi_0^*\Psi_0\right) \overline{\delta\hat{\chi}\delta\hat{\chi}'}. \tag{S-9}$$

Thus the quantum and classical fluctuations lead to the same density corrections whenever

$$\overline{\delta\chi\delta\chi'} = \langle \Delta\hat{\chi}\Delta\hat{\chi}' \rangle . \tag{S-10}$$

CALCULATION OF QUANTUM FLUCTUATIONS OF THE BREATHER'S PARAMETERS

Functions $f_\chi(x,t) = \partial(\sqrt{N}\Psi_0)/\partial\chi$, which are derivatives of the solution of the GP equation (S-1), multiplied by $\sqrt{N}$, are c-number solutions of the linearized NLS equation (2). One may introduce a conservation relation [30, 31]

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} \left[(\mathrm{Re}f_\chi)(\mathrm{Re}\bar{f}_\chi) + (\mathrm{Im}f_\chi)(\mathrm{Im}\bar{f}_\chi)\right] \mathrm{d}x = 0, \tag{S-11}$$

where $\bar{f}_\chi$ is the solution of the equation adjoint to Eq. (2),

$$i\frac{\partial\bar{f}_\chi}{\partial t} = -\frac{1}{2}\frac{\partial^2\bar{f}_\chi}{\partial x^2} - 2|\Psi_0|^2\bar{f}_\chi + \Psi_0^2\bar{f}_\chi^*. \tag{S-12}$$

Solution $\bar{f}_\chi$ is related to f_χ as $\bar{f}_\chi(x,t) = if_\chi(x,t)$. These adjoint functions fulfill orthogonality conditions

$$C_{\chi\xi} = \prec \bar{f}_\chi | f_\xi \succ, \tag{S-13}$$

with the quasi-inner product defined as

$$\prec \bar{f}_\chi | f_\xi \succ = \int_{-\infty}^{+\infty} \mathrm{d}x \left[(\mathrm{Re}\ \bar{f}_\chi)(\mathrm{Re}\ f_\xi) + (\mathrm{Im}\ \bar{f}_\chi)(\mathrm{Im}\ f_\xi)\right] \tag{S-14}$$

for derivatives f_χ and f_ξ. The derivatives of the two-soliton Gordon solution (S-2) with respect to 8 parameters N, Θ, V, B, n, θ, v, and b, as well as integrals in Eq. (S-14), can be calculated analytically. The only non-zero quasi-inner products of the derivatives at $t = 0$ are

$$C_{N\Theta} = -C_{\Theta N} = \frac{1}{2}, \tag{S-15a}$$
$$C_{VR} = -C_{RV} = \frac{N}{2}, \tag{S-15b}$$
$$C_{n\theta} = -C_{\theta n} = 1/4, \tag{S-15c}$$
$$C_{vb} = -C_{bv} = \frac{3N}{32}, \tag{S-15d}$$

with all other inner products vanishing. Thus, the parameters can be grouped to four canonically conjugate pairs, namely, $(N,\,\Theta)$, $(V,\,B)$, $(n,\,\theta)$, and $(v,\,b)$. Only the inner products corresponding to the canonically conjugate parameters χ and $\tilde{\chi}$ do not vanish, and $C_{\chi\tilde{\chi}}=-C_{\tilde{\chi}\chi}$.

Using the quantum-correction expansion (3), we find the expression for the initial ($t=0$) quantum fluctuations of the parameters:

$$\Delta\hat{\chi}_0 = C_{\tilde{\chi}\chi}^{-1} \prec \bar{f}_{\tilde{\chi}}(x,0)|\delta\hat{\psi}(x,0)\succ, \tag{S-16}$$

where the "real" and "imaginary" parts of the operators in the quasi-inner products (S-14) are defined as Re $\delta\hat{\psi}=(\delta\hat{\psi}+\delta\hat{\psi}^\dagger)/2$, Im $\delta\hat{\psi}=-i(\delta\hat{\psi}-\delta\hat{\psi}^\dagger)/2$. Similarly, quantum fluctuations may be defined for $t>0$ as [31, 33]

$$\Delta\hat{\chi}(t) = C_{\tilde{\chi}\chi}^{-1} \prec \bar{f}_{\tilde{\chi}}(x,0)|e^{-7iN^2t/128}\delta\hat{\psi}(x,t)\succ, \tag{S-17}$$

where the exponential factor cancels the mean-field phase shift. Using expansion (3)of operator $\hat{\psi}(x,t)$, we can express the quantum fluctuations at $t>0$ in terms of the initial quantum fluctuations as

$$\Delta\hat{\chi}(t) = \sum_{\xi} M_{\chi\xi}\Delta\hat{\xi}_0, \tag{S-18}$$

where

$$M_{\chi\xi} = C_{\tilde{\chi}\chi}^{-1} \prec \bar{f}_{\tilde{\chi}}(x,0)|e^{-7iN^2t/128} f_{\xi}(x,t)\succ, \tag{S-19}$$

while the integrals here are calculated numerically.

We find variances of the quantum fluctuations by using the "white noise" vacuum states, and expectation values of the creation/annihilation operators products calculated via Eq. (4). Substituting the definition of the quasi-inner product (S-14) in Eq. (S-16), we can express the quantum-fluctuation operators in an explicit form,

$$\Delta\hat{\chi}_0 = \frac{i}{2C_{\tilde{\chi}\chi}}\int_{-\infty}^{\infty}\mathrm{d}x\left(f_{\tilde{\chi}}(x,0)\delta\hat{\psi}^\dagger(x,0) - f_{\tilde{\chi}}^*(x,0)\delta\hat{\psi}(x,0)\right). \tag{S-20}$$

Note that operators $\Delta\hat{\chi}_0$ are Hermitian, i.e., $\Delta\hat{\chi}_0^\dagger=\Delta\hat{\chi}_0$. Then relations (4)lead to the average over vacuum of the products of the fluctuation operators,

$$\langle\Delta\hat{\chi}_0\Delta\hat{\xi}_0\rangle = \frac{1}{4C_{\tilde{\chi}\chi}C_{\tilde{\xi}\xi}}\int_{-\infty}^{\infty}\mathrm{d}x f_{\tilde{\chi}}^*(x,0)f_{\tilde{\xi}}(x,0). \tag{S-21}$$

Analytical integration leads then to the initial values of the quantum fluctuation variances of the breather's parameters which are given in Tables I and II.

Finally, one can show that the evolution of quantum fluctuations variances can be calculated as

$$\langle\Delta\hat{\chi}^2(t)\rangle = \sum_{\xi}|M_{\chi\xi}|^2\langle\Delta\hat{\xi}_0^2\rangle \tag{S-22}$$

[see Eq. (S-19)].

CALCULATION OF EXPECTATION VALUES IN THE CORRECTED VACUUM

Consider the quantum field of the mother soliton, $\hat{\Phi}(x)=\sqrt{N}\Phi_0(x)+\delta\hat{\psi}_{\mathrm{cont}}+\delta\hat{\psi}_{\mathrm{discr}}$, taken at $t=0$. The soliton solution is given by

$$\Phi_0(x) = \frac{1}{2\hbar}\sqrt{m|g_0|N}\,\mathrm{sech}\left(\frac{m|g_0|N}{2\hbar^2}(x-B)\right)\exp(i\frac{m}{\hbar}Vt+i\Theta). \tag{S-23}$$

It depends on the same COM parameters N, Θ, V, and B as the two-soliton solution (S-2). However, as the mother soliton is a pre-quench solution, we here take $|g_0|=g/4$. The continuum part can be expressed as [28, 44]

$$\delta\hat{\psi}_{\mathrm{cont}} = \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\left(U_k(x)\hat{b}_k+V_k^*(x)\hat{b}_k^\dagger\right), \tag{S-24}$$

with

$$U_k(x) = \frac{1 + (K^2-1)\cosh^2 X + 2iK\sinh X\cosh X}{(K-i)^2\cosh^2 X} e^{iKX}, \tag{S-25a}$$

$$V_k(x) = \frac{1}{(K-i)^2\cosh^2 X} e^{iKX}, \tag{S-25b}$$

where

$$X = \frac{m|g_0|N}{2\hbar^2}x, \quad K = \frac{2\hbar^2}{m|g_0|N}k \tag{S-26}$$

and operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ obey the standard bosonic commutation relations, $[\hat{b}_k, \hat{b}_{k'}^\dagger] = 2\pi\delta(k-k')$. The expectation values of the mother-soliton's continuum-fluctuations operators can be calculated and expressed as

$$\langle \delta\hat{\psi}(x)\delta\hat{\psi}(x')\rangle = \frac{1}{4\pi\xi}\left(-\frac{1}{2}\pi\mathrm{sech}^2(X)\mathrm{sech}^2(X')e^{-|X-X'|}(\cosh(2X)\,|X-X'| + (X-X')\sinh(2X) - 1)\right) \tag{S-27a}$$
$$+ \langle \delta\hat{\psi}_{discr}(x)\delta\hat{\psi}_{discr}(x')\rangle,$$
$$\langle \delta\hat{\psi}^\dagger(x)\delta\hat{\psi}^\dagger(x')\rangle = \langle \delta\hat{\psi}(x')\delta\hat{\psi}(x)\rangle^*, \tag{S-27b}$$
$$\langle \delta\hat{\psi}^\dagger(x)\delta\hat{\psi}(x')\rangle = \frac{1}{4\pi\xi}\left(\frac{1}{2}\pi\mathrm{sech}^2(X)\mathrm{sech}^2(X')e^{-|X-X'|}(|X-X'|+1)\right) + \langle \delta\hat{\psi}^\dagger_{\mathrm{discr}}(x)\delta\hat{\psi}_{\mathrm{discr}}(x')\rangle, \tag{S-27c}$$
$$\langle \delta\hat{\psi}(x)\delta\hat{\psi}^\dagger(x')\rangle = \delta(x-x') + \langle \delta\hat{\psi}^\dagger(x)\delta\hat{\psi}(x')\rangle. \tag{S-27d}$$

As in this case all of the products of fluctuation operators $\delta\hat{\psi}$ and $\delta\hat{\psi}^\dagger$ yield nonzero contributions, variances of parameter operators can be expressed as

$$\langle \Delta\hat{\chi}_0^2\rangle = -\frac{1}{4|C_{\tilde{\chi}X}|^2}\int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}x' \Big(f_{\tilde{\chi}}(x,0)f_{\tilde{\chi}}(x',0)\langle \delta\hat{\psi}^\dagger(x)\delta\hat{\psi}^\dagger(x')\rangle - f_{\tilde{\chi}}(x,0)f^*_{\tilde{\chi}}(x',0)\langle \delta\hat{\psi}^\dagger(x)\delta\hat{\psi}(x')\rangle \tag{S-28}$$
$$- f^*_{\tilde{\chi}}(x,0)f_{\tilde{\chi}}(x',0)\langle \delta\hat{\psi}(x)\delta\hat{\psi}^\dagger(x')\rangle + f^*_{\tilde{\chi}}(x,0)f^*_{\tilde{\chi}}(x',0)\langle \delta\hat{\psi}(x)\delta\hat{\psi}(x')\rangle\Big).$$

The fluctuations of the mother-soliton's discrete parameters are represented, like in Eq. (3), in terms of derivatives of the soliton's mean field:

$$\delta\hat{\psi}_{\mathrm{discr}}(x) = \sum_{\xi\in\{N,\Theta,V,B\}} \frac{\partial\Phi_0(x)}{\partial\xi}\Delta\hat{\xi}_0 = \sum_{\xi\in\{N,\Theta,V,B\}} f_\xi(x,0)\Delta\hat{\xi}_0. \tag{S-29}$$

where the last equality is a consequence of matching of the mean fields before and after the quench, $\Phi_0(x) = \Psi_0(x,0)$. Therefore, the contribution of $\delta\hat{\psi}_{\mathrm{discr}}(x)$ to variances (S-28) of the relative parameters becomes proportional to the inner products $C_{\chi\xi}$ [see (S-13)] with $\xi \in \{N,\Theta,V,B\}$ and $\chi \in \{n,\theta,v,b\}$. Then, $C_{\chi\xi} = 0$ as ξ and χ are not canonically conjugate [see Eq. (S-15)], therefore fluctuations of the discrete parameters of the mother soliton do not contribute to variances (S-27) of the breather's relative parameters. Equations (S-27) are used to calculate the corrected vacuum average in the main text.